APPLICATION OF SEMI-EMPIRICAL PROBABILITY DISTRIBUTIONS IN WAVE-STRUCTURE INTERACTION PROBLEMS

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Wave-Structure Interaction

Extreme Environment


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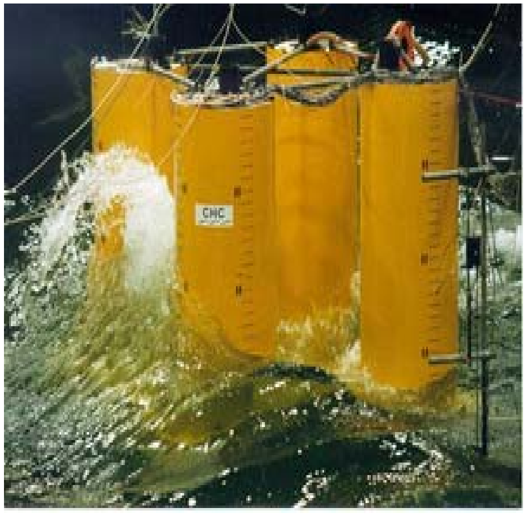

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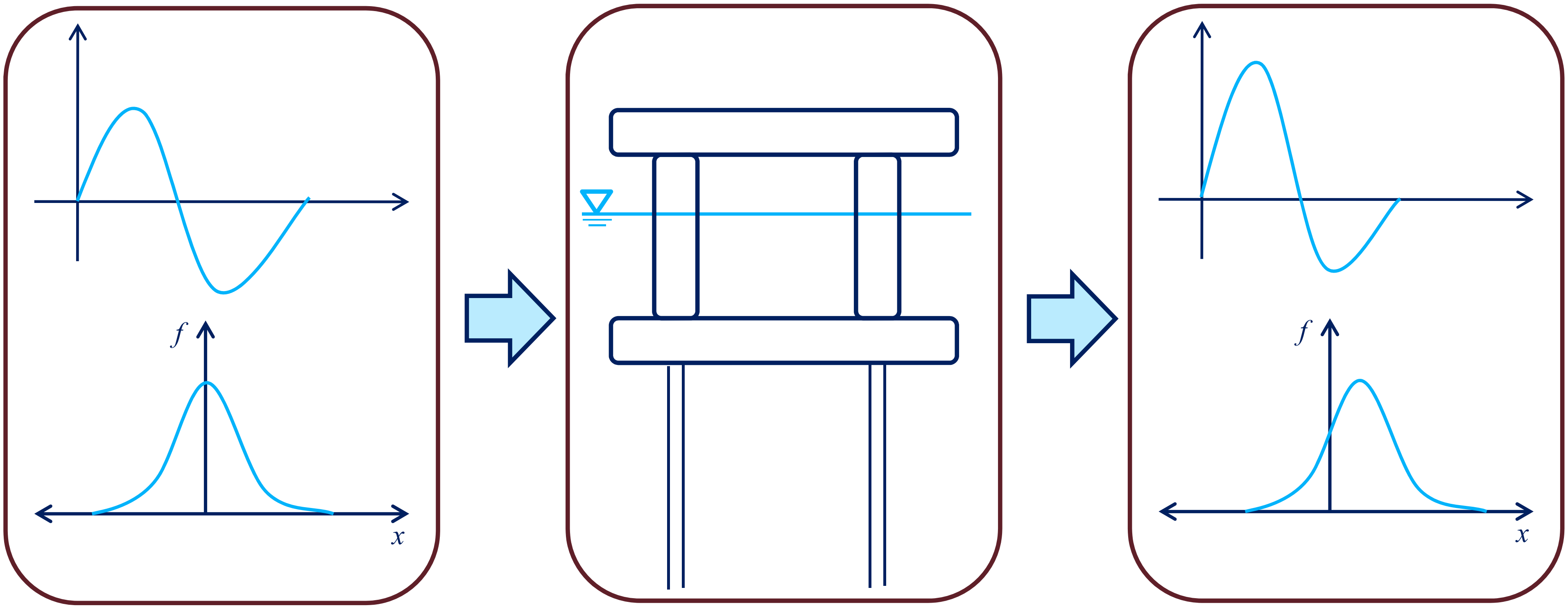
Wave Structure Interaction

Input Interaction Output



Incident random waves Structure Random response

Parametric Probability Distributions

- Structural Form
- Parameters

Example:

3-parameter Weibull Distribution

Probability Density Function (PDF):

$$f(x) = \frac{\delta}{\kappa}\left(\frac{x-\gamma}{\kappa}\right)^{\delta-1} \exp\left(-\left(\frac{x-\gamma}{\kappa}\right)^{\delta}\right)$$

Cumulative distribution Function (CDF):

$$F(x) = 1 - \exp\left(-\left(\frac{x-\gamma}{\kappa}\right)^{\delta}\right)$$

Quantile function:

$$x(u) = \gamma + \kappa\left[-\ln(1-u)\right]^{1/\delta}$$

Probability Distribution Estimation Approaches

Theoretical Approach

Structural form:
Obtained from a mathematical model.
Parameters:
Obtained from a mathematical model.

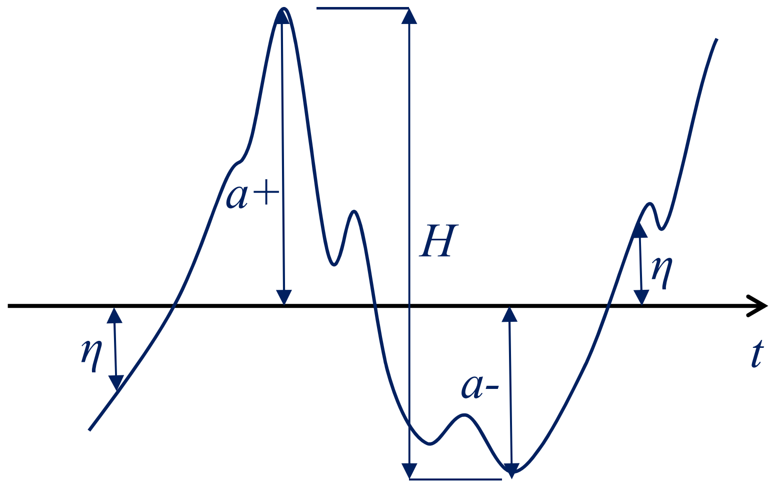
Empirical Approach

Structural form:
Selected from standard probability distributions.
Parameters:
Estimated from sample data statistics.

Semi-Empirical Approach

Structural form:
Obtained from a mathematical model.
Parameters:
Estimated from sample data statistics.

Weakly Non-linear Random Variable



Random Variable	**Linear Process**	**Semi-Empirical model**
Crest a^+, Trough a^- Height H	Rayleigh Distribution	Quadratic Rayleigh Distribution

Quadratic Rayleigh: Model Development

Probability distribution of crests in a linear narrow-banded random process (Rayleigh)

$$f_{\zeta}(x) = \frac{x}{R}\exp\left(-\frac{x^2}{2R}\right), \qquad x > 0$$

$$\zeta_n = \gamma + \alpha\zeta + \beta\zeta^2 \qquad \alpha > 0$$

Random variable transformation rule

$$f_{\zeta_n}(x) = \frac{\chi-\alpha}{2\beta R\chi}\exp\left(-\frac{(\chi-\alpha)^2}{8R\beta^2}\right), \qquad x > \gamma$$

$$F_{\zeta_n}(x) = 1-\exp\left(-\frac{(\chi-\alpha)^2}{8R\beta^2}\right), \qquad \chi = \left(\alpha^2 + 4\beta(x-\gamma)\right)^{1/2}$$

$$x_{\zeta_n}(u) = \gamma - 2\beta R\ln(1-u) + \alpha\left(-2R\ln(1-u)\right)^{1/2}.$$

Quadratic Rayleigh: Extreme Analysis

Theory of ordered statistics:

ζ_{max} maximum in N crests ζ_n

$$f_{\zeta_{max}}(x) = N f_{\zeta_n}(x)\left[F_{\zeta_n}(x)\right]^{N-1}$$

$$F_{\zeta_{max}}(x) = \left[F_{\zeta_n}(x)\right]^{N}$$

Asymptotic form:

$$\lim_{N\to\infty}\left[F_{\zeta_n}(x)\right]^N = F_{\zeta_{max}}\left(\frac{x-a_N}{b_N}\right)$$

For large number of waves

Gumbel distribution

$$F_{\zeta_{max}}(x) = \exp\left(-\exp\left(-\frac{x-a_N}{b_N}\right)\right)$$

with parameters

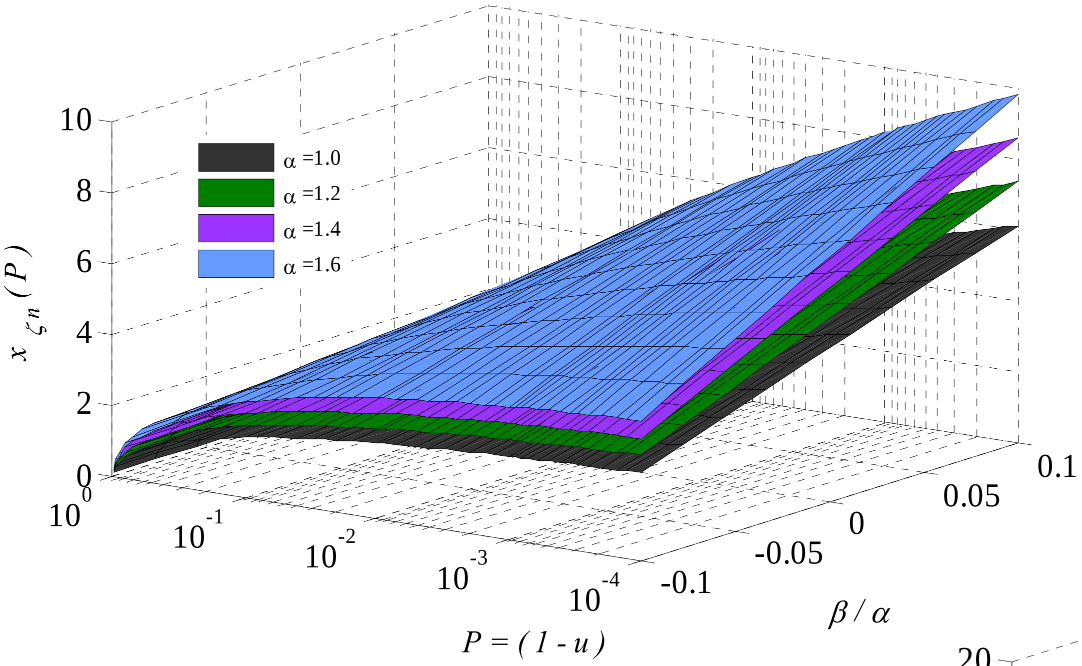
$$a_N = x_{\zeta_n}\left(1-\frac{1}{N}\right)$$

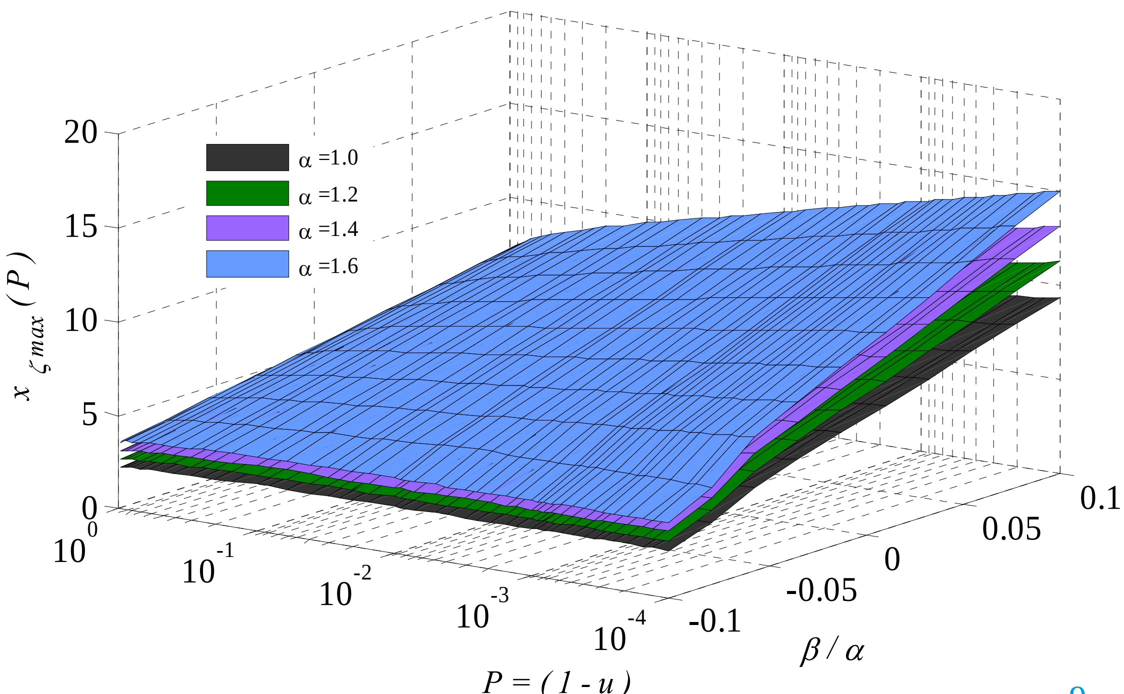
$$b_N = x_{\zeta_n}\left(1-\frac{1}{Ne}\right) - a_N$$

$$a_N = \gamma + 2\beta R\ln(N) + \alpha\left(2R\ln(N)\right)^{1/2}$$

$$b_N = 2\beta R + \alpha(R)^{1/2}\left(2\ln(N)\right)^{-1/2}$$

Quadratic Rayleigh: Distributions





Quadratic Rayleigh: Empirical Parameter Estimation

Method of Moments

Equating the first three distribution moments with their corresponding sample moments.

Distribution Moments:

$$\mu_1(X) = \int_0^1 x(u)\,du \qquad n=1$$

$$\mu_n(X) = \int_0^1 \left(x(u) - \mu_1\right)^n du \qquad n>1$$

Sample Moments:

$$\hat{\mu}_1(X) = \frac{1}{N}\sum_{i=1}^{N} x_i$$

$$\hat{\mu}_n(X) = \frac{1}{N}\sum_{i=1}^{N}\left(x_i - \hat{\mu}_1\right)^n$$

Method of L-moments

Equating the first three distribution L-moments with their corresponding sample L-moments.

Distribution L-Moments:

$$\lambda_n(X) = \int_0^1 x(u)\,P_{n-1}^*(u)\,du, \qquad P_n^*(u) = \sum_{k=0}^{r} p_{n,k}^*\, u^k$$

$$p_{n,k}^* = \frac{(-1)^{n-k}(n+k)!}{(k!)^2(n-k)!}$$

Sample L-Moments:

$$l_{n+1}(X) = \sum_{k=0}^{n} p_{n,k}^*\,\hat{\mu}_{PW,2,k}(X)$$

$$\hat{\mu}_{PW,2,n}(X) = N^{-1}\binom{N-1}{n}^{-1}\sum_{j=n+1}^{N}\binom{j-1}{n} x_{j:N}$$

Quadratic Rayleigh: Empirical Parameter Estimation

Method of Moments

$$\mu_1(\zeta_n) = \gamma + 2\beta R + \alpha(2R)^{1/2}\Gamma(3/2),$$

$$\mu_2(\zeta_n) = 4\beta^2 R^2 + \alpha\beta(2R)^{3/2}\Gamma(3/2) + 2\alpha^2 R(1-\pi/4),$$

$$\mu_3(\zeta_n) = 16\beta^3 R^3 + 9(2)^{1/2}\alpha\beta^2 R^{5/2}\Gamma(3/2) + 12\alpha^2\beta R^2\left(1-\Gamma^2(3/2)\right) + \alpha^3(2R)^{3/2}\left(2\Gamma^3(3/2) - \frac{3}{2}\Gamma(3/2)\right).$$

Method of L-moments

$$\lambda_1(\zeta_n) = \gamma + 2\beta R + \alpha(2R)^{1/2}\Gamma(3/2),$$

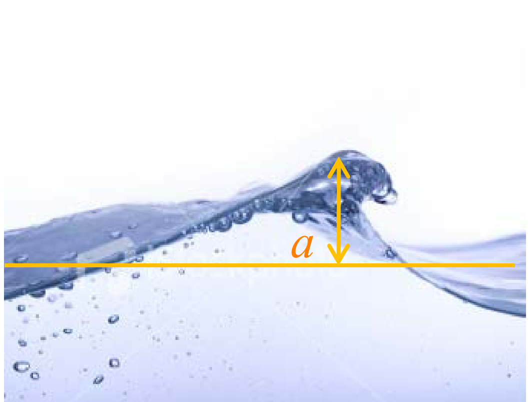
$$\lambda_2(\zeta_n) = \beta R + \alpha(R)^{1/2}\left(2^{1/2}-1\right)\Gamma(3/2),$$

$$\lambda_3(\zeta_n) = \alpha(R)^{1/2}\left(2^{1/2} - 3 + (8/3)^{1/2}\right)\Gamma(3/2) + \frac{\beta R}{3}$$

Example 1: Weakly Non-linear Wave Crests

Quadratic Rayleigh model justification:

http://learn.uci.edu



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Second-order stokes theory for Narrow-banded waves:

Surface elevation:

$$\eta(x,t) = a\cos(k_0 x - \omega_0 t + \varepsilon) + \frac{k_0 a^2}{2}\cos 2(k_0 x - \omega_0 t + \varepsilon)$$

Wave crests:

$$a_n = a + \frac{k_0}{2}a^2 \quad \xrightarrow{\&} \quad f_a(x) = x\exp(-x^2/2) \longrightarrow \text{One-parameter Rayleigh-Stokes}$$

Semi-empirical models:

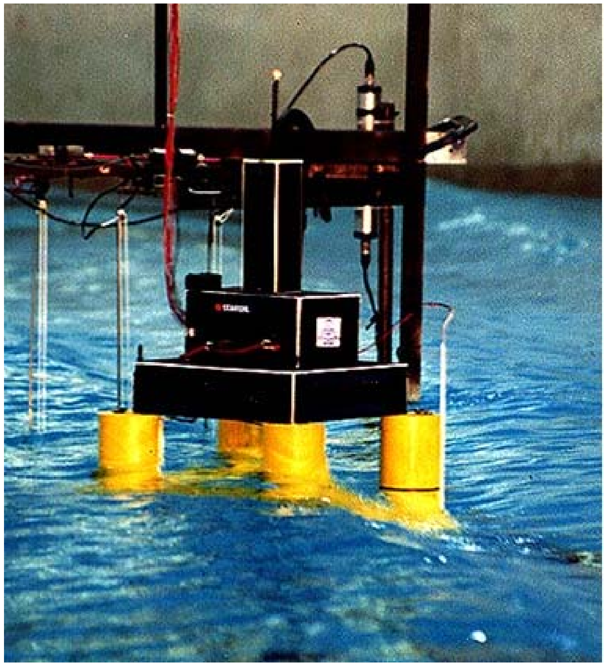
$$\zeta_n = \gamma + \alpha\zeta + \beta\zeta^2$$

$$\longrightarrow f_\zeta(x) = (x/R)\exp(-x^2/2R) \longrightarrow \text{Quadratic Rayleigh}$$

Example 1: Model Test Data

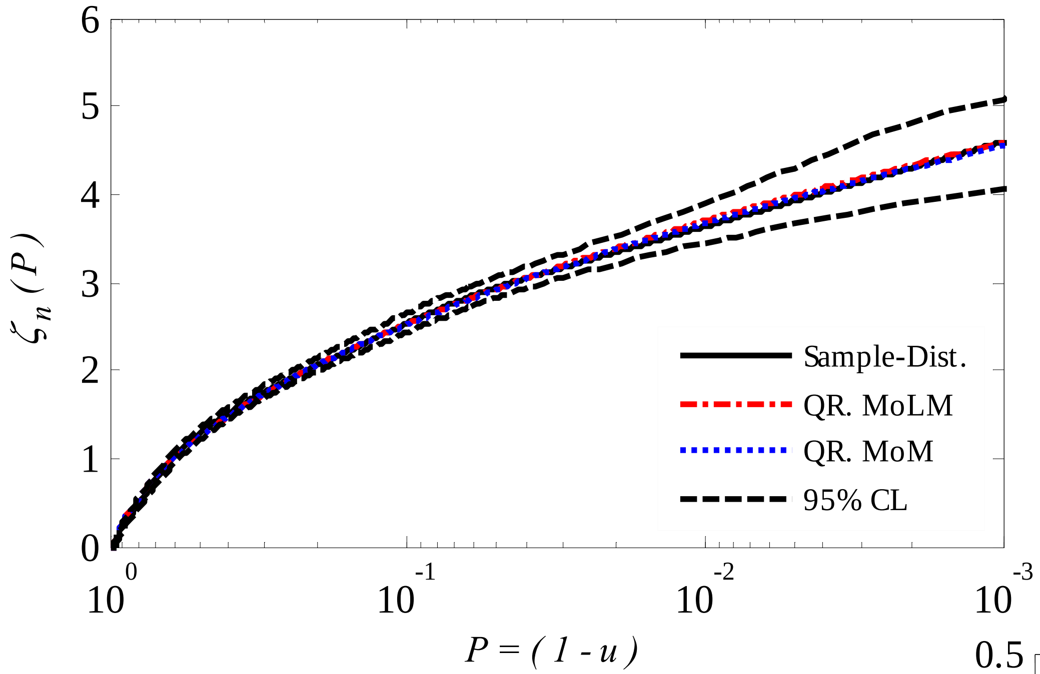
Draft	28.50 m
Column Diameter	8.75 m
Column Spacing	28.50 m
Pontoon Height	6.25 m
Pontoon Width	6.25 m

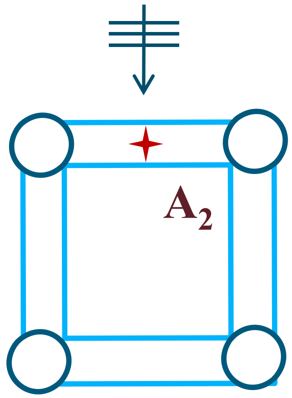
Design Storm	**Description**	**H_s (m)**	**T_p (sec)**	**γ_s**
S_1	100 yr West Africa	4.0	16.0	2.0

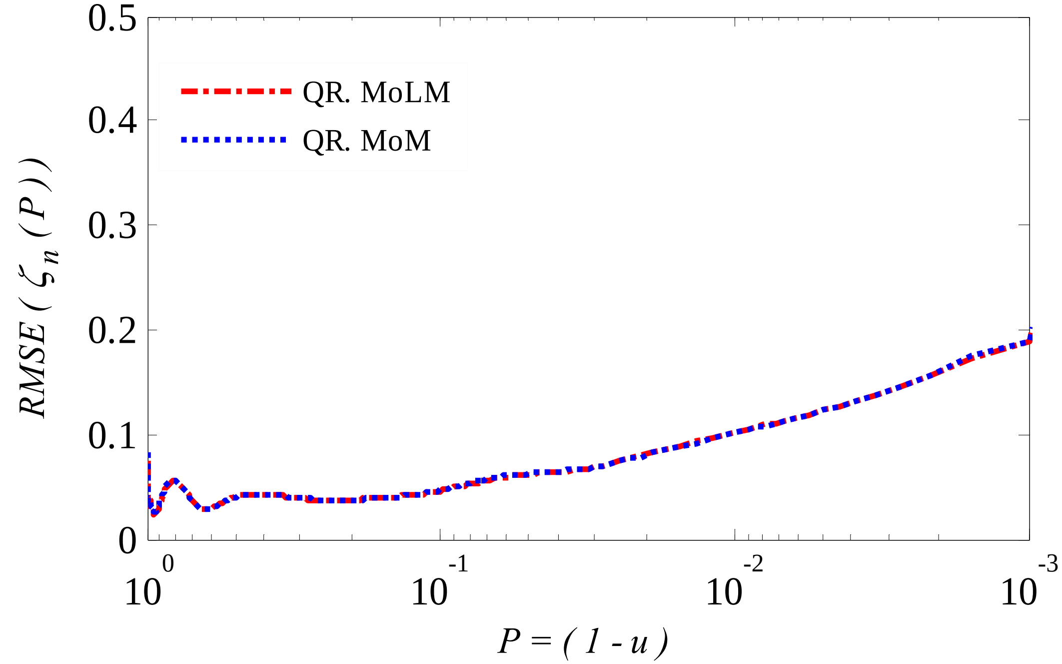


Courtesy of OTRC

Example 1: Sample Results







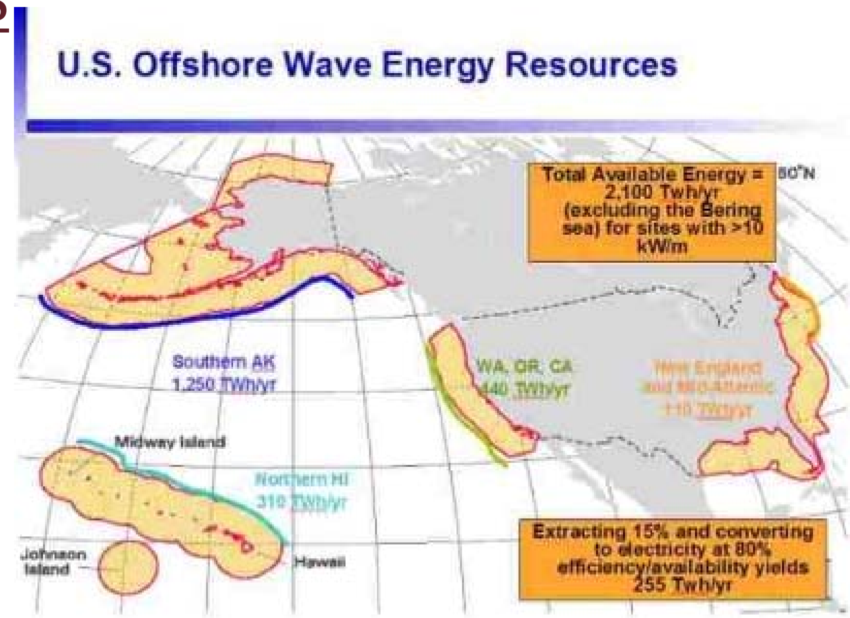
Example 2: Wave Power Resources

Background:

- long term mean wave power

$$\overline{P}_{d,S} = \int_0^\infty \frac{\rho g^2}{4\pi f} G_{\eta\eta}(f)\, df = \frac{\rho g^2}{64\pi} T_{-1} H_s^2$$

$$\Rightarrow \quad \overline{P}_d = \sum_i \overline{P}^i_{d,S} F(S_i)$$



EPRI Report

Empirical model:

- Based on an empirical joint probability distribution of wave height and period (Myrhaug et. al.).

Theoretical model:

- Based on the theoretical joint probability distribution of wave height and period and linear wave theory (Izadparast and Niedzwecki).

Example 2: Wave Power Resources

Quadratic Rayleigh model justification:

	Instantaneous Wave power		Mean Wave power		Normalized Wave power
Deep water:	$P_d = \frac{\rho g^2}{32\pi} T H^2$	÷	$\bar{P}_d = \frac{\rho g^2}{64\pi} T_{-1} H_s^2$	=	$p_d = \frac{P_d}{\bar{P}_d} = \beta \zeta^2$
Shallow water:	$P_s = \frac{\rho g^{3/2}}{8} d^{1/2} H^2$	÷	$\bar{P}_s = \frac{\rho g^{3/2}}{16} d^{1/2} H_s^2$	=	$p_s = \frac{P_s}{\bar{P}_s} = \beta \zeta^2$

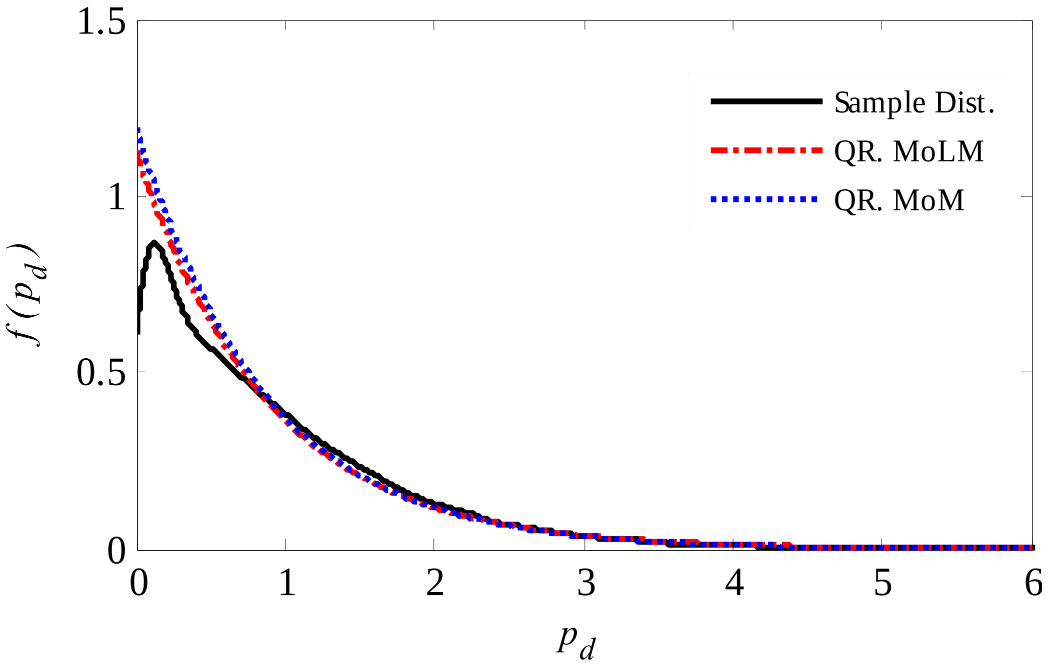
Semi-Empirical Model:

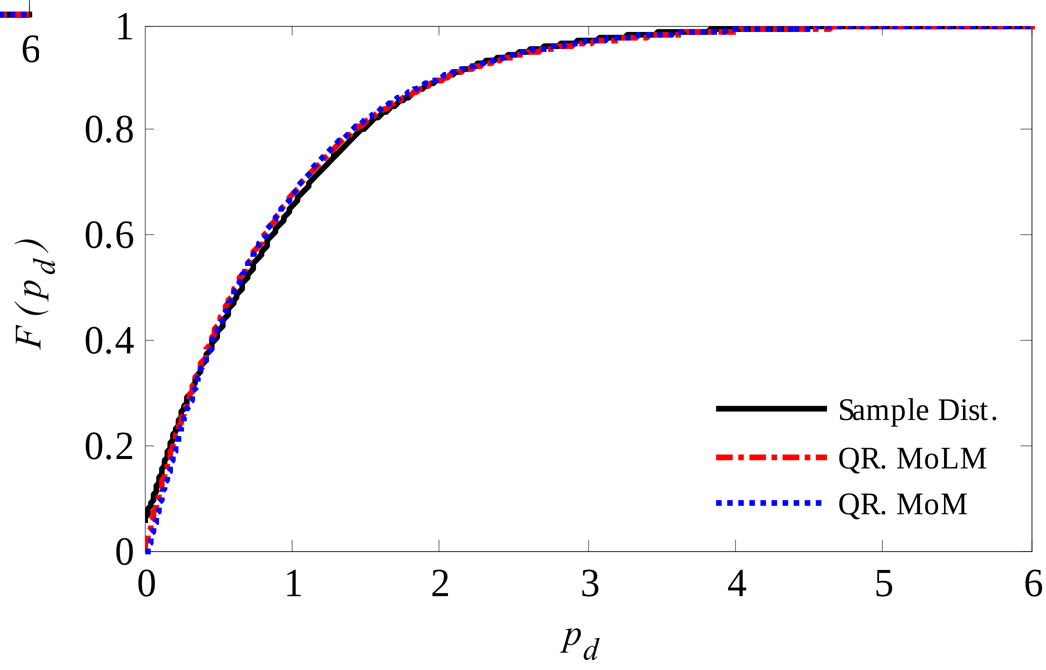
$$\zeta_n = \gamma + \beta \zeta^2$$

$$f_\zeta(x) = (x/R)\exp\left(-x^2/2R\right)$$

$$R = 4.0$$

Example 2: Wave Power Resources





Concluding Remarks

- Semi-empirical models are introduced as a data analysis tool for estimation of probability distribution of non-linear random variables.
- Semi-empirical models could:
  - Improve the flexibility of theoretical model in capturing the probability distribution of data,
  - Provide some insight about the physical process.
- Development of Quadratic Rayleigh distribution is discussed:
  - Model structure development,
  - Extreme statistics,
  - Parameter estimation.
- As examples, the application of Quadratic Rayleigh distribution on probability distribution estimation of two random variables, i.e. weakly non-linear wave crests, and wave power, were studied.

Thank You!

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Weakly Non-linear Random Variable

Perturbation expansion of weakly non-linear variable:

$$\zeta_n = \varepsilon^0 \zeta^{(0)} + \varepsilon^1 \zeta^{(1)} + \varepsilon^2 \zeta^{(2)} + \cdots \longrightarrow \text{Stokes Expansion}$$

$$\zeta_n = \varepsilon^0 \delta_0 \zeta^0 + \varepsilon^1 \delta_1 \zeta^1 + \varepsilon^2 \delta_2 \zeta^2 + \cdots$$

$$\zeta_n = \gamma + \alpha \zeta + \beta \zeta^2 \qquad \alpha \ge 0, \qquad |\beta| \ll \alpha$$

Random variable transformation rule ⟵

$$f_{\zeta_n}(x) = f_{\zeta}\left(\zeta = G(\zeta_n), \zeta_n = x\right) \left| \frac{\partial}{\partial \zeta_n}\left(G(\zeta_n), \zeta_n = x\right) \right|$$

$$G(\zeta_n) = \frac{-\alpha \pm \chi}{2\beta} \quad \text{and} \quad \chi = \left(\alpha^2 + 4\beta(\zeta_n - \gamma)\right)^{1/2}$$