Empirical Moment-Based Estimation of Rayleigh-Stokes Distribution Parameters

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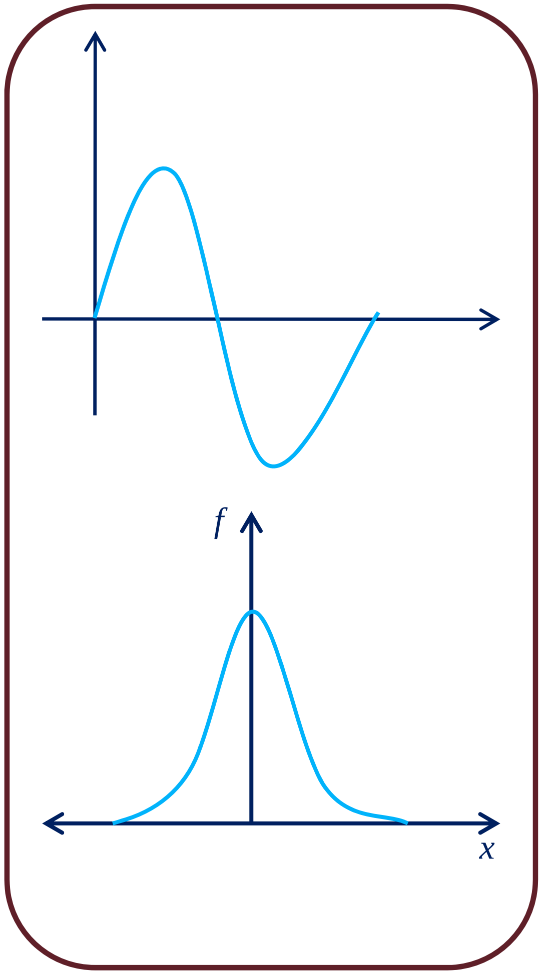
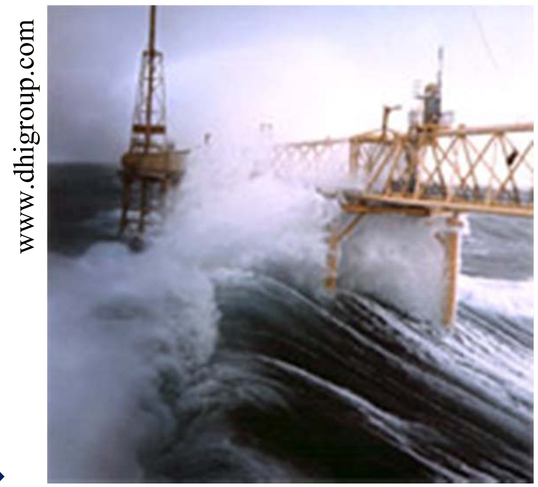
Wave-Structure Interaction

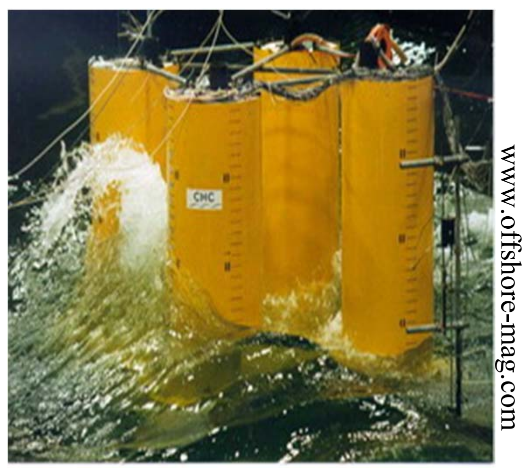
Input

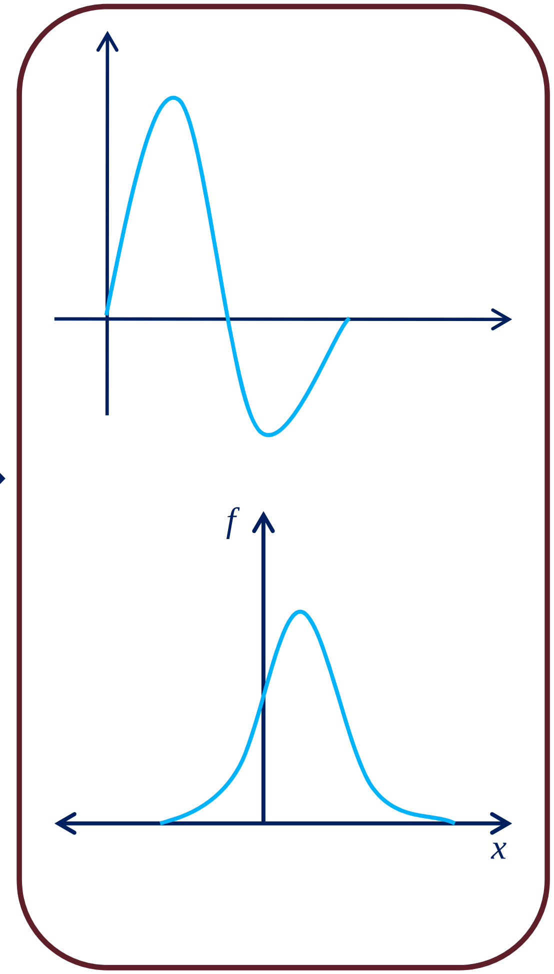
Interaction

Output









Incident random waves

Structure

Random response

Motivation

- It is necessary to include non-linear terms in the approximations,
- Robust models are required that can accurately estimate the probability distribution of non-linear random variables,
- It is important to address the physics correctly,
- It is vital to obtain reliable estimates of the extreme statistics,
- Data are available: from full scale measurements, model test data, and calibrated numerical models,

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Rayleigh-Stokes Model Development

PDF of crests in a linear narrow-banded random process (Rayleigh)

$$f_{\zeta}(x) = x\exp\left(-x^2/2\right) \qquad x>0$$

$$\zeta_n = \gamma + \alpha\zeta + \beta\zeta^2 \qquad \alpha > 0$$

$\beta > 0$

$$f_{\zeta_n}(x) = \frac{\chi-\alpha}{2\beta\chi}\exp\left(-\frac{(\chi-\alpha)^2}{8\beta^2}\right)$$

$$F_{\zeta_n}(x) = 1-\exp\left(-\frac{(\chi-\alpha)^2}{8\beta^2}\right)$$

$$x_{\zeta_n}(u) = \gamma - 2\beta\ln(1-u) + \alpha\left(-2\ln(1-u)\right)^{1/2}$$

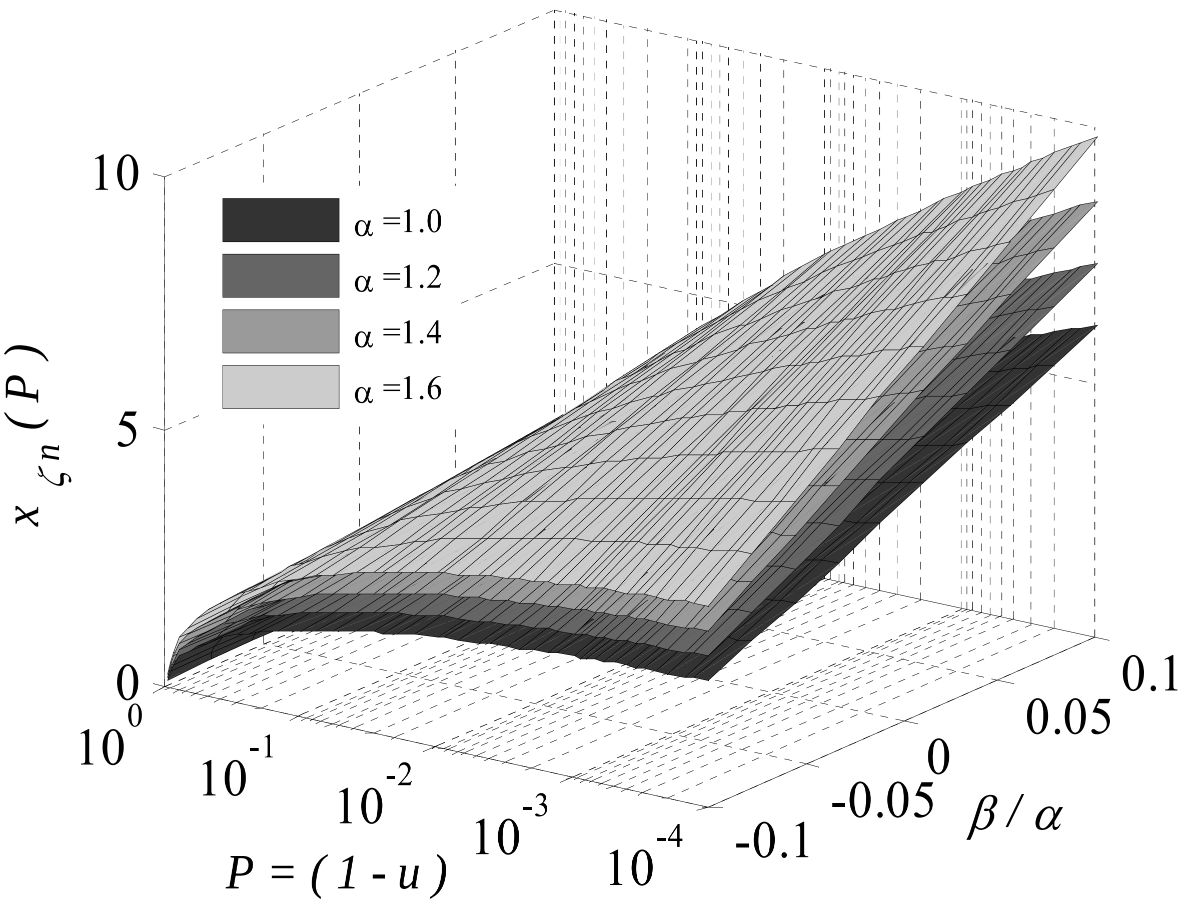
$$\chi = \left(\alpha^2 + 4\beta(x-\gamma)\right)^{1/2}$$

$\beta < 0$

$$f_{\zeta_n}(x) = \frac{1}{2\beta\chi}\left[(\chi-\alpha)\exp\left(-\frac{(\chi-\alpha)^2}{8\beta^2}\right)H_\gamma(x) + -(\chi+\alpha)\exp\left(-\frac{(\chi+\alpha)^2}{8\beta^2}\right)\right]$$

$$F_{\zeta_n}(x) = \left[1-\exp\left(-\frac{(\chi-\alpha)^2}{8\beta^2}\right)\right]H_\gamma(x) + \exp\left(-\frac{(\chi+\alpha)^2}{8\beta^2}\right)$$

Rayleigh-Stokes Quantile Distribution



$$x_{\zeta_n}(u) = \gamma - 2\beta \ln(1-u) + \alpha \left(-2\ln(1-u)\right)^{1/2}$$

Extreme Analysis

Theory of ordered statistics:

$\zeta_{\max}$ maximum in N crests ζ_n ⇨

$$f_{\zeta_{\max}}(x) = N\, f_{\zeta_n}(x)\left[F_{\zeta_n}(x)\right]^{N-1}$$

$$F_{\zeta_{\max}}(x) = \left[F_{\zeta_n}(x)\right]^{N}$$

Asymptotic form:

$$\lim_{N\to\infty}\left[F_{\zeta_n}(x)\right]^N = F_{\zeta_{\max}}\left(\frac{x-a_N}{b_N}\right)$$

For large number of waves

⇩ Gumbel distribution

$$F_{\zeta_{\max}}(x) = \exp\left(-\exp\left(-\frac{x-a_N}{b_N}\right)\right)$$

with parameters

$$a_N = x_{\zeta_n}\left(1-\frac{1}{N}\right)$$

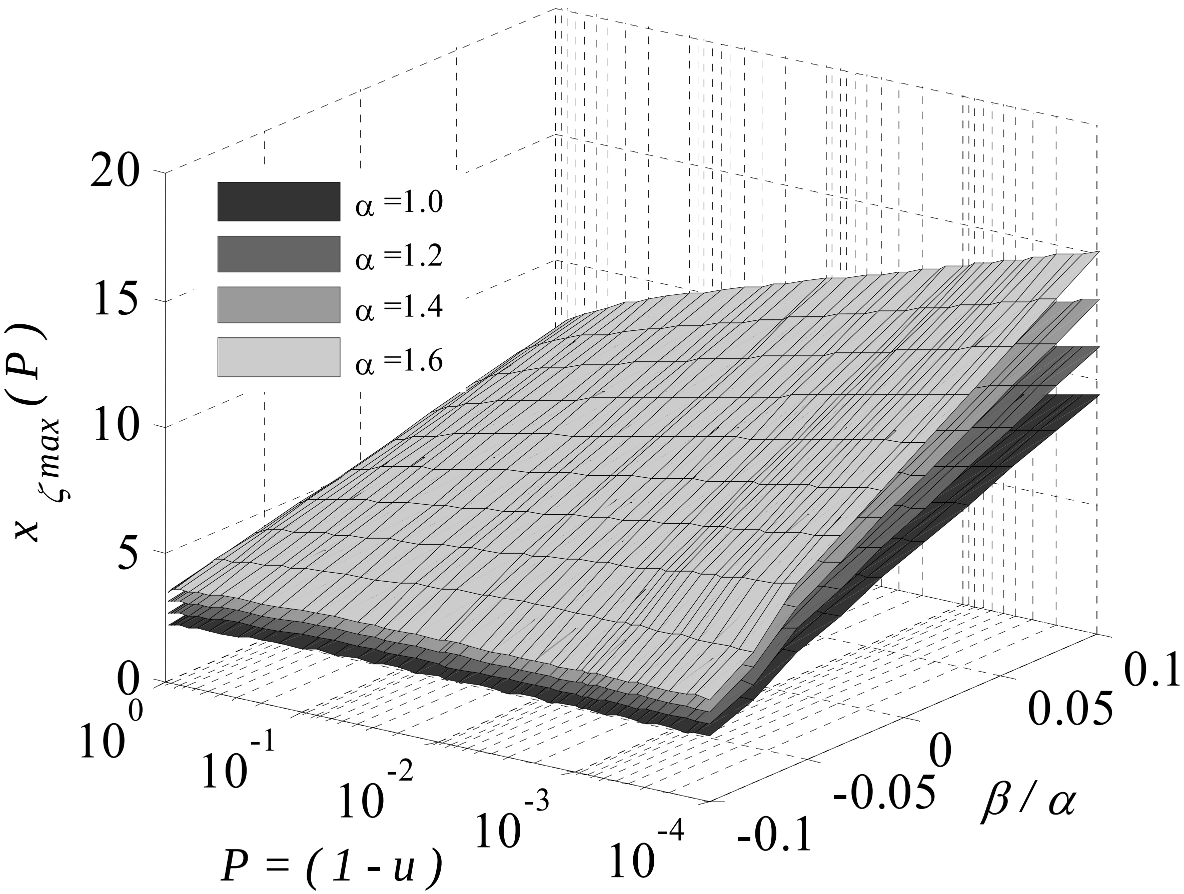
$$b_N = x_{\zeta_n}\left(1-\frac{1}{Ne}\right) - a_N$$

⇨

$$a_N = \gamma + 2\beta \ln(N) + \alpha\,(2\ln(N))^{1/2}$$

$$b_N = 2\beta + \alpha\,(2\ln(N))^{-1/2}$$

Quantile Distribution of Rayleigh-Stokes Maxima



$$x_{\zeta_{\max}}(u) = a_N - b_N \ln(-\ln(u))$$

$$N = 1000$$

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Empirical Parameter Estimation

Method of Moments

Equating the first three distribution moments with their corresponding sample moments.

Distribution Moments:

$$\mu_1(X) = \int_0^1 x(u)\,du \qquad n=1$$

$$\mu_n(X) = \int_0^1 \left(x(u)-\mu_1\right)^n du \qquad n>1$$

Sample Moments:

$$\hat{\mu}_1(X) = \frac{1}{N}\sum_{i=1}^{N} x_i$$

$$\hat{\mu}_n(X) = \frac{1}{N}\sum_{i=1}^{N}\left(x_i-\hat{\mu}_1\right)^n$$

Method of L-moments

Equating the first three distribution L-moments with their corresponding sample L-moments.

Distribution L-Moments:

$$\lambda_n(X) = \int_0^1 x(u)P^*_{n-1}(u)\,du, \qquad P^*_n(u) = \sum_{k=0}^{r} p^*_{n,k}\,u^k$$

$$p^*_{n,k} = \frac{(-1)^{n-k}(n+k)!}{(k!)^2(n-k)!}$$

Sample L-Moments:

$$l_{n+1}(X) = \sum_{k=0}^{n} p^*_{n,k}\hat{\mu}_{PW,2,k}(X)$$

$$\hat{\mu}_{PW,2,n}(X) = N^{-1}\binom{N-1}{n}^{-1}\sum_{j=n+1}^{N}\binom{j-1}{n}x_{j:N}$$

Empirical Parameter Estimation

Method of Moments

$$\mu_1(\zeta_n) = \gamma + 2\beta + \alpha(2)^{1/2}\,\Gamma(3/2),$$

$$\mu_2(\zeta_n) = 4\beta^2 + \alpha\,\beta(2)^{3/2}\,\Gamma(3/2) + 2\alpha^2(1-\pi/4),$$

$$\mu_3(\zeta_n) = 16\beta^3 + 9(2)^{1/2}\,\alpha\beta^2\;\Gamma(3/2) + 12\alpha^2\beta\left(1-\Gamma^2(3/2)\right) + \alpha^3(2)^{3/2}\left(2\Gamma^3(3/2) - \frac{3}{2}\Gamma(3/2)\right).$$

Method of L-moments

$$\lambda_1(\zeta_n) = \gamma + 2\beta + \alpha(2)^{1/2}\,\Gamma(3/2),$$

$$\lambda_2(\zeta_n) = \beta + \alpha\left(2^{1/2} - 1\right)\Gamma(3/2),$$

$$\lambda_3(\zeta_n) = \alpha\left(2^{1/2} - 3 + (8/3)^{1/2}\right)\Gamma(3/2) + \frac{\beta}{3}$$

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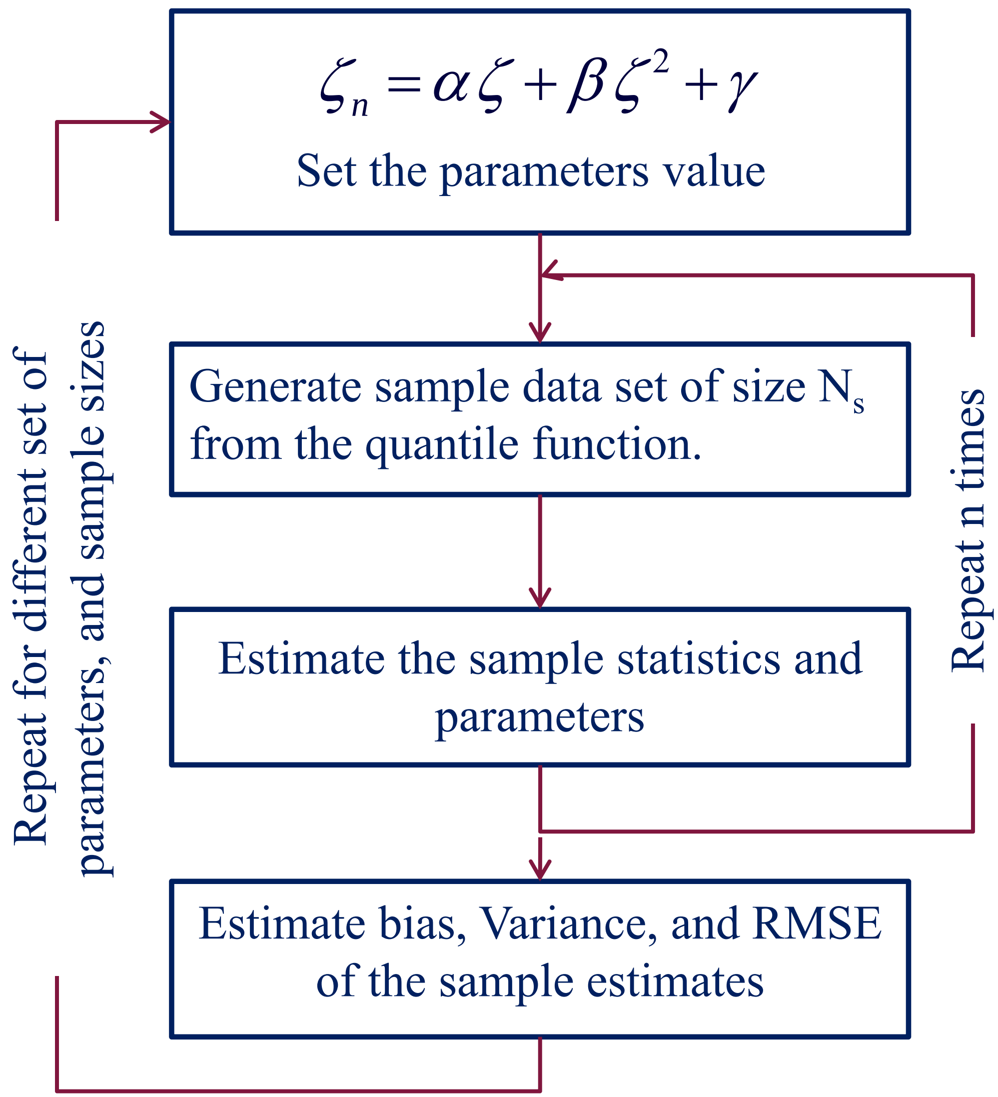
Empirical Parameter Estimation Uncertainty Analysis



Studied Parameters:

$$1.0 \le \alpha \le 1.6$$

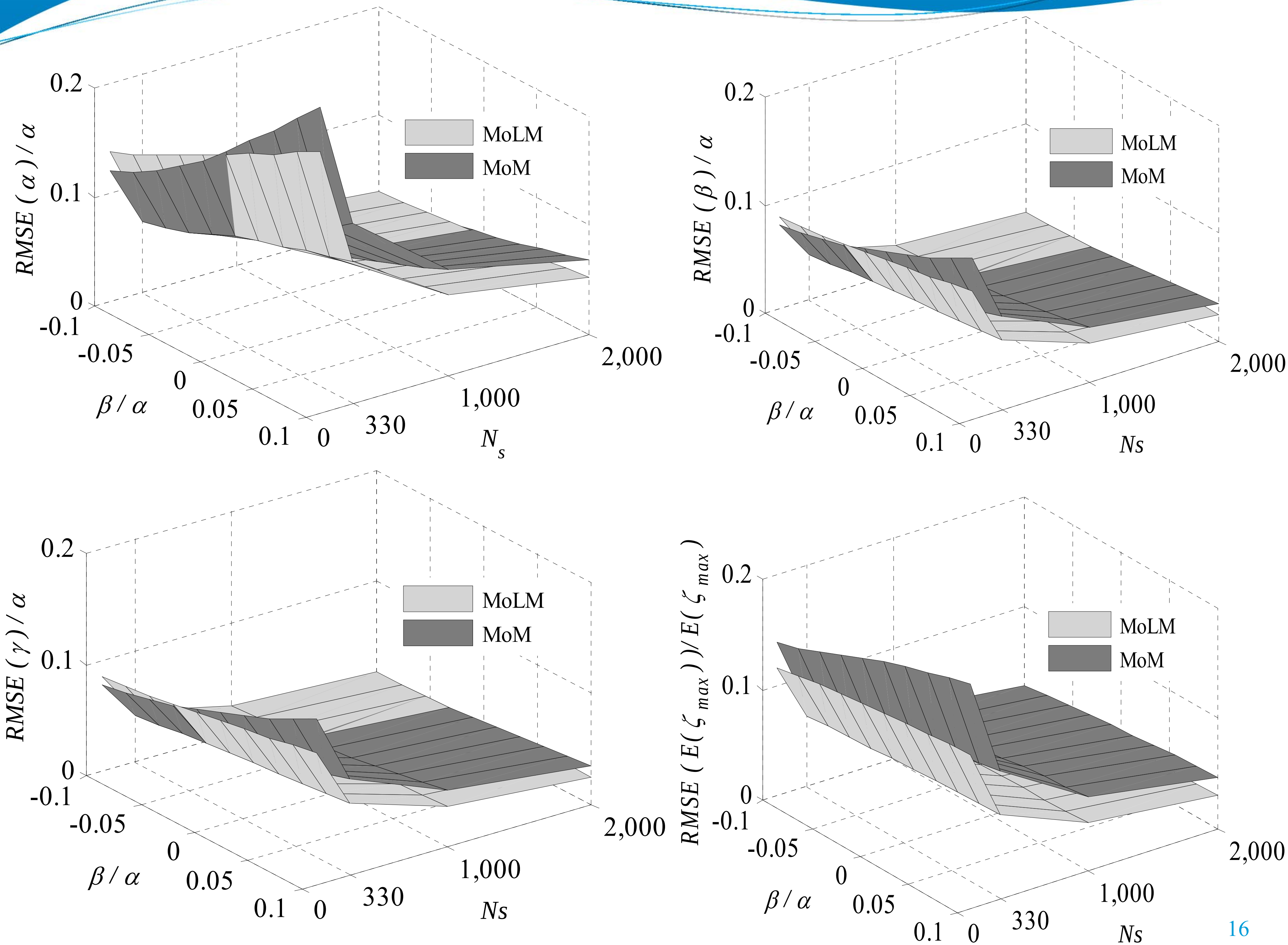
$$-\alpha/10 \le \beta \le \alpha/10$$

$$\gamma = 0.0$$

$N_s = 2{,}000$; $1{,}000$; 330; 100

Measure of Performance:

$$RMSE(\hat{\varphi}) = \left(E\left[(\varphi - \hat{\varphi})^2\right]\right)^{1/2}$$



RMSE (α) / α
0.2
0.1
0
-0.1
-0.05
0
0.05
0.1
β / α
0
330
1,000
2,000
N_s
MoLM
MoM
RMSE (β) / α
0.2
0.1
0
-0.1
-0.05
0
0.05
0.1
β / α
0
330
1,000
2,000
Ns
MoLM
MoM
RMSE (γ) / α
0.2
0.1
0
-0.1
-0.05
0
0.05
0.1
β / α
0
330
1,000
2,000
Ns
MoLM
MoM
RMSE (E(ζ max))/ E(ζ max)
0.2
0.1
0
-0.1
-0.05
0
0.05
0.1
β / α
0
330
1,000
2,000
Ns
MoLM
MoM

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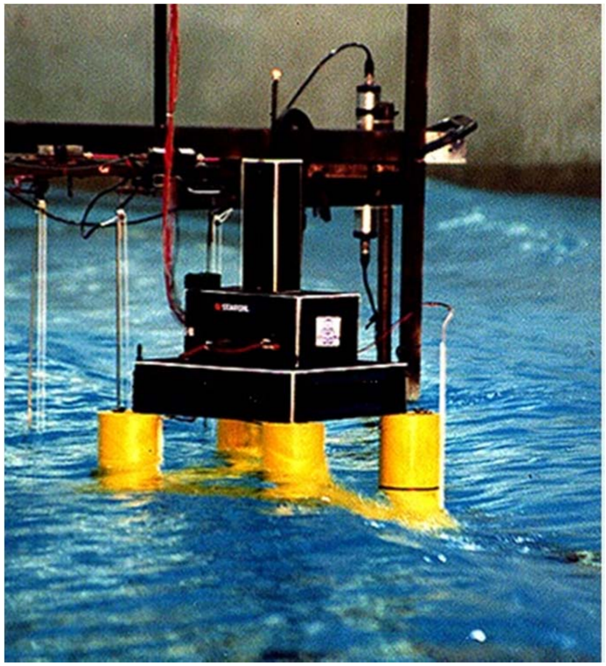
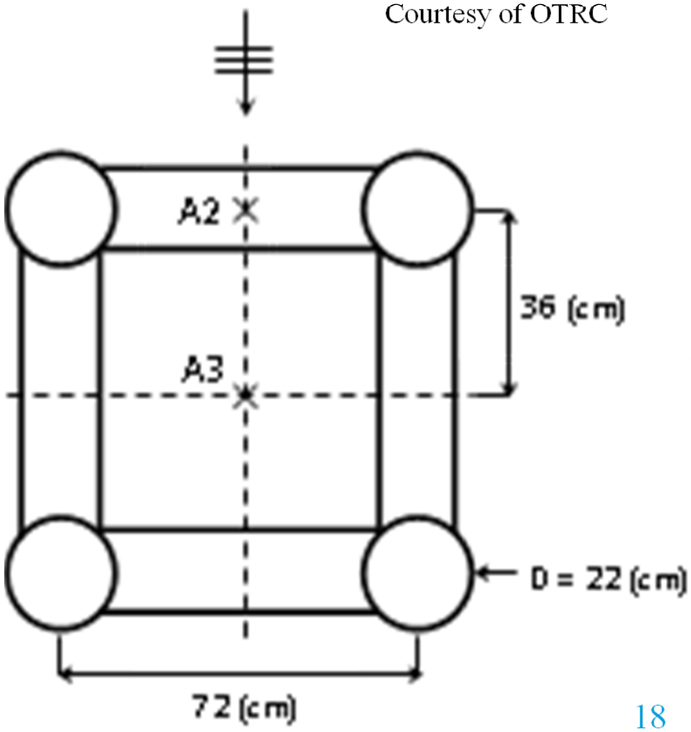
Model Test Data

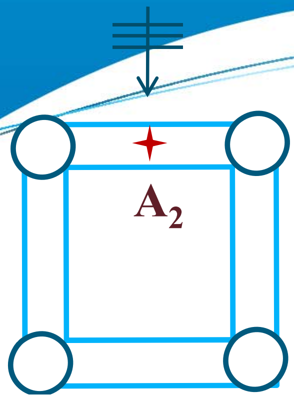
Draft	28.50 m
Column Diameter	8.75 m
Column Spacing	28.50 m
Pontoon Height	6.25 m
Pontoon Width	6.25 m
Scale	1/40
Water Depth	668 m

Description	H_s (m)	T_p (sec)	γ_s
100 yr West Africa	4.0	16.0	2.0



Courtesy of OTRC



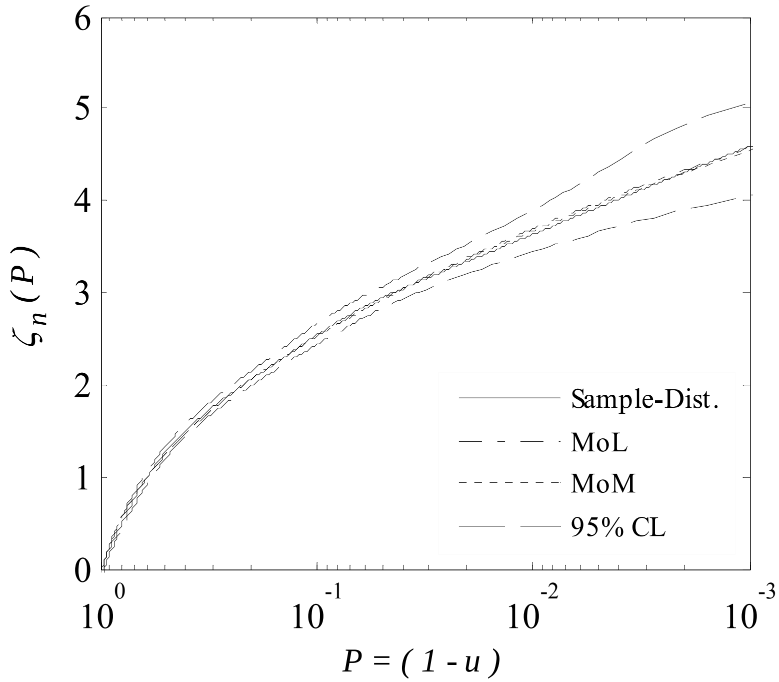


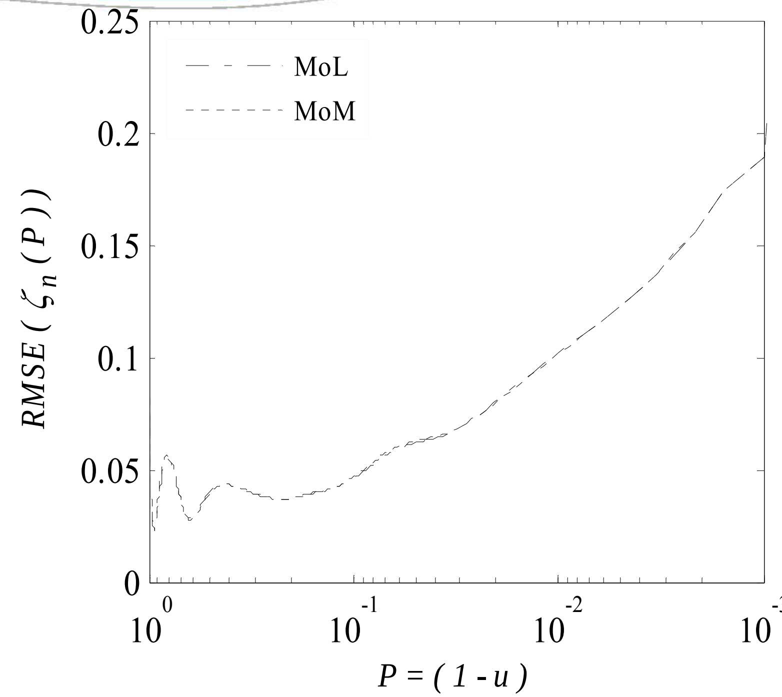
MoLM

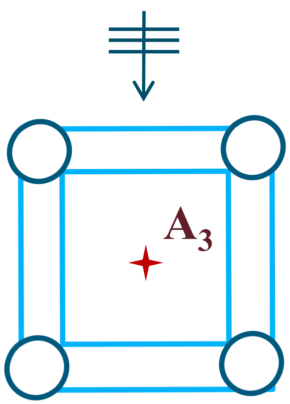
E (ζ_{max}) = 4.76 (m)

MoM

E (ζ_{max}) = 4.72 (m)





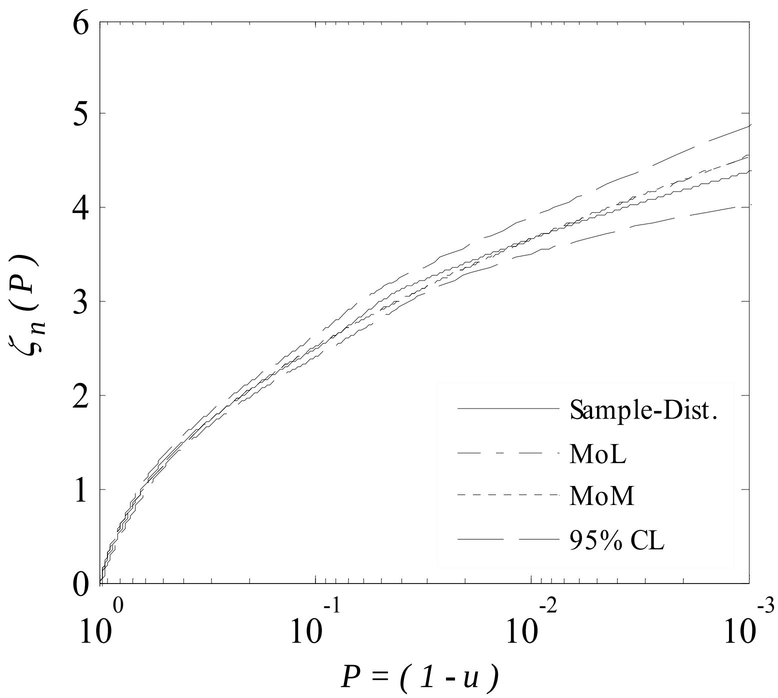


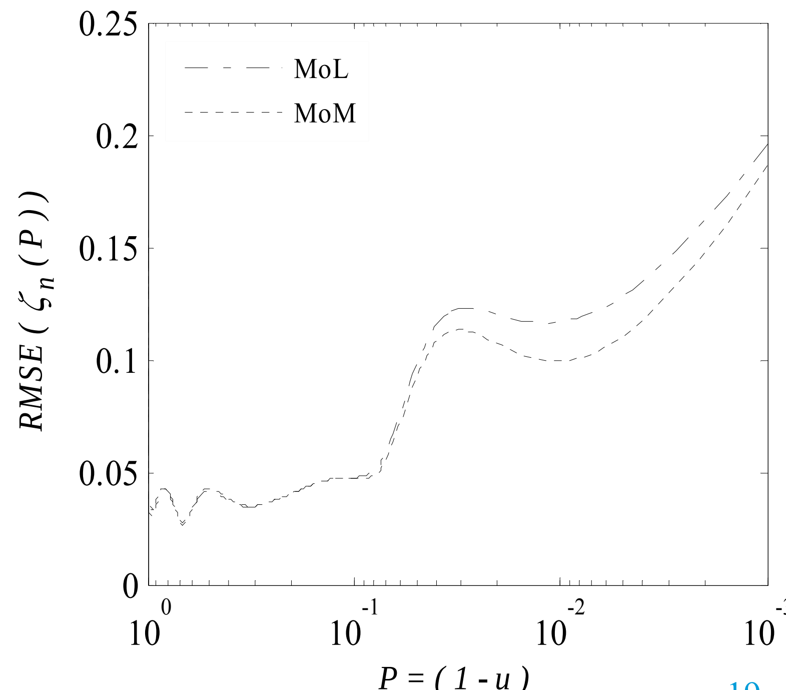
MoLM

E (ζ_{max}) = 4.72 (m)

MoM

E (ζ_{max}) = 4.73 (m)





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Concluding Remarks

- The three-parameter Rayleigh-Stokes model is studied as a semi-empirical probability distribution of weakly non-linear crests and troughs.
- The model parameters are estimated from application of method of moments (MoM) and method of L-moments (MoLM).
- The application of Rayleigh-Stokes model for extreme analysis and estimation of extreme statistics is discussed.
- It is shown that the tail distribution is highly sensitive to the variation of the model parameters specially to the variability of the non-linear term.
- It is observed that MoLM and MoM perform similarly well for large samples while MoLM is the better alternative in case of small samples and samples with contaminated observations.
- MoLM found to be more robust in the case of positive non-linear term while MoM performs better in the case of negative non-linear term.

Concluding Remarks

- The extreme statistics of empirical Rayleigh-Stokes model estimated from small samples $N_s<330$ should be used with care.
- The practical example studied here, once more, showed that Rayleigh-Stokes model is considerably successful in representing the probability distribution of non-linear wave crests.

Acknowledgments

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Extreme crests may not be that bad after all!